Transmuted New Generalized Inverse Weibull Distribution

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Abstract

This paper introduces the transmuted new generalized inverse Weibull distribution by using the quadratic rank transmutation map (QRTM) scheme studied by Shaw et al. (2007). The proposed model contains twenty three lifetime distributions as special sub-models. Some mathematical properties of the new distribution are formulated, such as quantile function, Rényi entropy, mean deviations, moments, moment generating function and order statistics. The method of maximum likelihood is used for estimating the model parameters. We illustrate the flexibility and potential usefulness of the new distribution by means of two real data sets.

Keywords: New generalized inverse Weibull distribution; Moment estimation; Moment generating function; Order statistics; Maximum likelihood estimation.

1. Introduction

In the theory of life testing, many different families of lifetime distribution have been developed for describing the reliability behaviour of the components or process. These new families of lifetime distributions can be obtained by adding parameters to the well established distributions for obtaining more flexibility in the new extended lifetime distribution. Because of this motivation, we introducing a new lifetime distribution called the transmuted new generalized inverse weibull distribution by using quadratic rank transmutation map (QRTM) technique studied by Shaw et al. (2007). Historically speaking, the Inverse Weibull distribution (also known as type 2 extreme value or the Fréchet distribution) is a very flexible lifetime distribution having the inverse Rayleigh and inverse exponential distributions as special sub-models commonly used for modelling reliability data. The Inverse Weibull distribution has been applied in many areas of scientific disciplines, such as reliability engineering, aeronautics, hydrology, physics, biomedical sciences, agriculture, pharmacutical sciences, psychology, metrology, economics and actuarial sciences etc. More recently, Khan and King (2016) proposed the new generalized inverse Weibull (NGIW) distribution and investigated many structural properties for modeling reliability engineering application.

The cdf of the NGIW distribution is given by

$$F(x) = 1 - \left[1 - exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}, \tag{1}$$

where $\beta, \phi > 0$ are the shape parameters and $\alpha, \gamma > 0$ are the scale parameters. The probability density function corresponding to (1) is given by

$$f(x) = \phi\left\{\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right\}\left(\frac{1}{x}\right)^{2} exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1 - exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}, \tag{2}$$

The CDF given in equation (1) approches to the eleven lifetime distributions when its parameters change. Khan and King (2012) proposed the modified Inverse Weibull distribution and presented a comprehensive description of the mathematical properties of this model along with its reliability behavior. Using quadratic rank transmutation map (QRTM) technique, we introduce the transmuted the new generalized inverse Weibull (NGIW) distribution by introducing a new parameter λ that would offer more flexibility in the proposed model. Several distributions have been proposed under this methodology such as transmuted extreme value distribution (Gokarna and Chris, 2009) studied with application to climate data, the transmuted Weibull distribution (Gokarna and Chris, 2011) proposed with two applications, Gokarna (2013) proposed the transmuted Log-Logistic distribution and studied its various structural properties. Khan and King (2013) proposed the transmuted modified Weibull distribution as an important competitive model with eleven lifetime distributions as sub-models along with its theoretical properties. Khan and King (2013) studied the flexibility of the transmuted generalized Inverse Weibull distribution with application to reliability data. Merovci (2013) studied the transmuted rayleigh distribution. Elbatal et al. (2013a, 2013b) proposed and studied the transmuted additive Weibull and transmuted modified inverse Weibull distributions. Khan et al. (2014a, 2014b) proposed the transmuted inverse Weibull distribution and studied its various structural properties with an application to survival data. More recently, Khan and King (2015) explored the flexibility of the transmuted modified Inverse Rayleigh distribution using QRTM technique which extends the modified Inverse Rayleigh distribution with application to reliability data. A random variable X is said to have transmuted distribution if its cumulative distribution function (cdf) is given by

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \leq 1 \tag{3}$$

and

$$f(x) = g(x)\{(1 + \lambda) - 2\lambda G(x)\}, \tag{4}$$

where $G(x)$ is the cdf of the baseline distribution. It is important to note that at λ = 0 we have the distribution of the baseline random variable.

The rest of this article is organized as follows, In Section 2, we present the analytical shapes of the probability density and distribution function of the proposed model. Some mathematical properties are formulated in Section 3, such as expressions for the moment estimation and moment generating function. Maximum likelihood estimates (MLEs) of the unknown parameters are discussed in Section 4. We derive expressions for the Rényi entropy and mean deviations in section 5. The order statistics are formulated in Section 6. In Section 7, we compare the proposed model with three other lifetime distributions by means of two real data sets to illustrate its usefulness. In Section 8, we offer some Concluding remarks.

2. Transmuted New Generalized Inverse Weibull Distribution

A random variable $\boldsymbol{X}$ is said to have transmuted new generalized inverse Weibull (TNGIW) distribution with parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\phi} > \mathbf{0}$ and $|\boldsymbol{\lambda}| \leq \mathbf{1}$, $\boldsymbol{x} > \mathbf{0}$. If the probability density function is given by

$$f(x) = \phi\left\{\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right\}\exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x), \quad (5)$$

$$\mathcal{U}(x) = \left(\frac{1}{x}\right)^{2}\left\{1 - \lambda + 2\lambda\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}. \quad (6)$$

The CDF corresponding to equation (5) is given by

$$F(x) = \left\{1 - \left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1 + \lambda\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}. \quad (7)$$

Figure 1 shows the visualizations of the transmuted new generalized inverse Weibull PDF with some selected choice of parameters. Some useful characterizations of the TNGIW distribution are formulated as reliability function (RF), hazard function and reversed hazard function defined as

$$R(x) = 1 - \left\{1 - \left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1 + \lambda\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}, \quad (8)$$

$$h(x) = \frac{\phi\left(\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)\exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x)}{1 - \left\{1 - \left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1 + \lambda\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}}, \quad (9)$$

and

$$r(x) = \frac{\phi\left(\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)\exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x)}{\left\{1 - \left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1 + \lambda\left[1 - \exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}}. \quad (10)$$

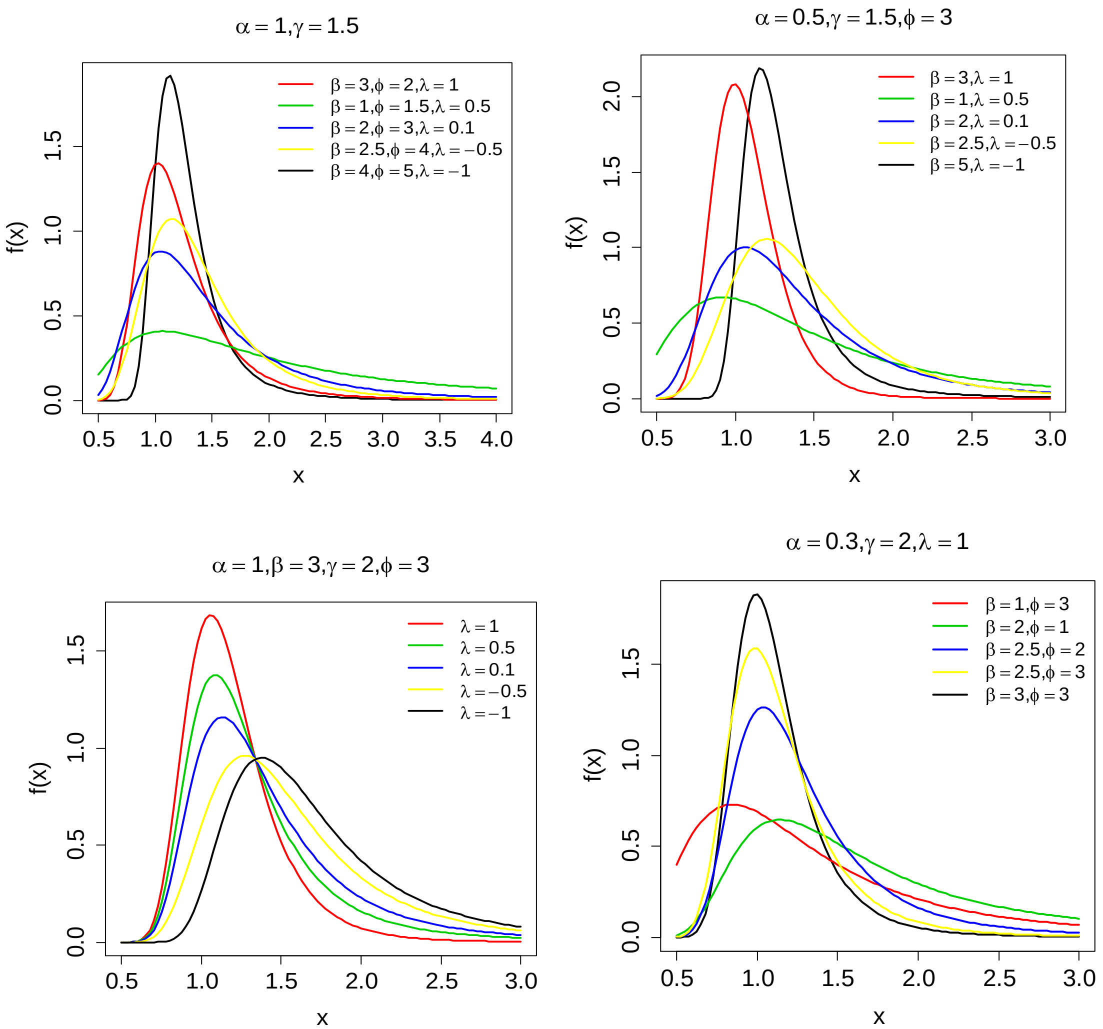

Figure 1: Plots of the TNGIW pdf for some parameter values.

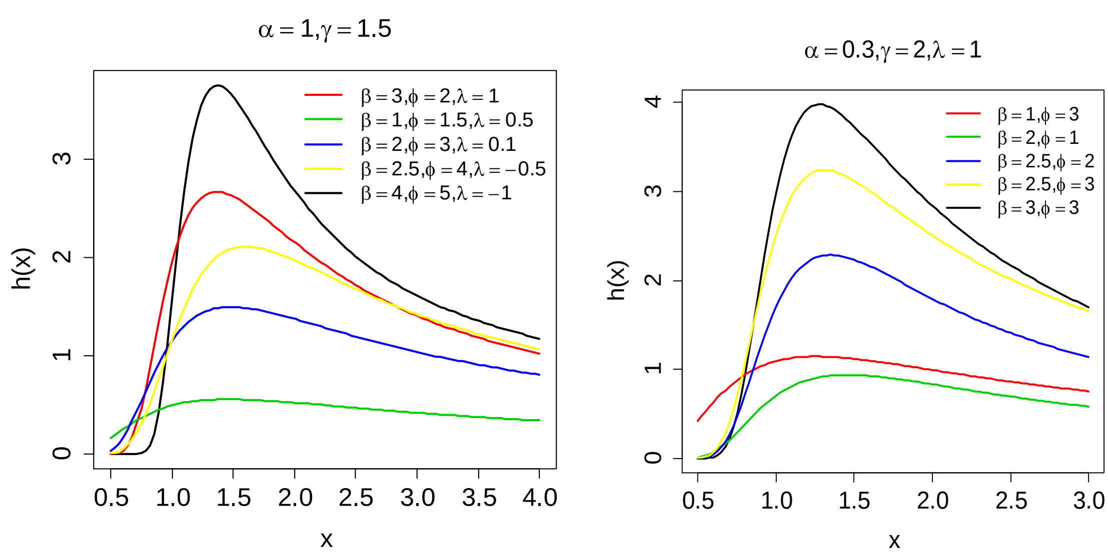

Figure 2: Plots of the TNGIW hf for some parameter values.

The cumulative hazard function (CHF) of the TNGIW distribution is defined as

$$H(x) = \int_0^x \frac{\phi\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x)}{1-\left\{1-\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1+\lambda\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}}\,dx,$$

$$H(x) = -\ln\left[1-\left\{1-\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\left\{1+\lambda\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}\right]. \quad (11)$$

The quantile function of the TNGIW distribution is the real solution of the following equation

$$\gamma\left(\frac{1}{x_q}\right)^{\beta}+\frac{\alpha}{x_q}+ln\left\{1-\left(1-\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right)^{\frac{1}{\phi}}\right\}=0. \quad (12)$$

A random variable X with density (5) is denoted by $X \sim TNGIW(x;\alpha,\beta,\gamma,\phi,\lambda)$. When the transmuting parameter $\lambda = 0$, we obtain the new generalized inverse Weibull distribution. Figure 2 illustrates the hazard function of the TNGIW distribution with different choice of parameters. These visualizations of the failure rates show that the proposed model has upside down hazard rate function for some selected choice of parameters. Table 1 listed twenty three lifetime distributions as special sub-models of the transmuted new generalized inverse Weibull distribution.

Table 1: Sub-models of the Transmuted New Generalized Inverse Weibull distribution

S. No	Model	α	β	γ	ϕ	λ	Authors
1	TNGIE	–	1	–	–	–	New
2	TNGIR	–	2	–	–	–	New
3	TMIW	–	–	–	1	–	Elbatal (2013b)
4	TMIR	–	2	–	1	–	Khan & King (2015)
5	TMIE	–	1	–	1	–	New
6	TGIW	0	–	–	–	–	Khan & King (2013b)
7	TGIR	0	2	–	–	–	New
8	TGIE	0	1	–	–	–	New
9	TIW	0	–	–	1	–	Khan & King (2014b)
10	TIR	0	2	–	1	–	Vikas et al. (2014)
11	TIE	0	1	–	1	–	Oguntunde et al. (2015)

12	NGIW	–	–	–	–	0	Khan & King (2016)
13	NGIR	–	2	–	–	0	New
14	NGIE	–	1	–	–	0	New
15	MIW	0	–	–	1	0	Khan & King (2012)
16	MIR	0	2	–	1	0	Khan M. S. (2014)
17	MIE	0	1	–	1	0	New
18	GIW	0	–	–	–	0	Gusmão et al. (2009)
19	GIR	0	2	–	–	0	Gusmão et al. (2009)
20	GIE	0	1	–	–	0	Gusmão et al. (2009)
21	IW	–	–	–	1	0	Khan et al. (2008)
22	IR	–	2	–	1	0	Voda, V. Gh. (1972)
23	IE	–	1	–	1	0	Klugman et al. (2012)

Note: T: Transmuted; G, Generalized; M, Modified; N, New; I, Inverse; W, Weibull; E, Exponential; R, Rayleigh.

3. Statistical Properties

This section formulates the k^{th} moment and the moment generating function of the transmuted new generalized inverse Weibull distribution.

Theorem 1: If X has the $TNGIW(x;\alpha,\beta,\gamma,\phi,\lambda)$ distribution with $|\lambda| \leq 1$, then the k^{th} moment of X say $\acute{\mu}_k$ is given as follows

$$\acute{\mu}_k = (1-\lambda)\sum_{i,j=0}^{\infty}\binom{\phi-1}{i}\frac{\gamma^{j}\phi(-1)^{i+j}(i+1)^{j}}{j!}\xi(\alpha,\beta,\gamma,i,j,k)$$
$$+2\lambda\sum_{i,j=0}^{\infty}\binom{2\phi-1}{i}\frac{\gamma^{j}\phi(-1)^{i+j}(i+1)^{j}}{j!}\xi(\alpha,\beta,\gamma,i,j,k),$$

where

$$\xi(\alpha,\beta,\gamma,i,j,k) = \frac{\alpha\Gamma(\beta j-k+1)}{\left(\alpha(i+1)\right)^{\beta j-k+1}} + \frac{\beta\gamma\Gamma(\beta(j+1)-k)}{\left(\alpha(i+1)\right)^{\beta(j+1)-k}}.$$

Proof: The k^{th} moment of X can be obtained from (5) as

$$\acute{\mu}_k = \int_0^{\infty} x^k\,\phi\left(\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1 - exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x)dx,$$

By using equation (6) we can write the above integral as

$\acute{\mu}_k =$

$$(1-\lambda)\int_0^\infty x^{k-2}\phi\left(\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right) exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1} dx$$
$$+2\lambda\int_0^\infty x^{k-2}\phi\left(\alpha + \beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right) exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{2\phi-1} dx,$$

the above integral reduces to

$$\acute{\mu}_k = (1-\lambda)\int_0^\infty x^{k-2}\phi\alpha\, exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1} dx +$$
$$(1-\lambda)\int_0^\infty x^{k-\beta-1}\phi\beta\gamma\, exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1} dx$$
$$+2\lambda\int_0^\infty x^{k-2}\phi\alpha\, exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{2\phi-1} dx$$
$$+2\lambda\int_0^\infty x^{k-\beta-1}\phi\beta\gamma exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{2\phi-1} dx,$$

By using the Binomial expansion, the above integral reduces to

$$\acute{\mu}_k = (1-\lambda)\sum_{i=0}^{\infty}\binom{\phi-1}{i}\phi\alpha(-1)^i \int_0^\infty x^{k-2}\, exp\left\{-\frac{\alpha}{x}(i+1)-\gamma\left(\frac{1}{x}\right)^{\beta}(i+1)\right\}dx$$
$$+(1-\lambda)\sum_{i=0}^{\infty}\binom{\phi-1}{i}\phi\beta\gamma(-1)^i \int_0^\infty x^{k-\beta-1}\, exp\left\{-\frac{\alpha}{x}(i+1) - \gamma\left(\frac{1}{x}\right)^{\beta}(i+1)\right\}dx$$
$$+2\lambda\sum_{i=0}^{\infty}\binom{2\phi-1}{i}\phi\alpha(-1)^i \int_0^\infty x^{k-2}\, exp\left\{-\frac{\alpha}{x}(i+1)-\gamma\left(\frac{1}{x}\right)^{\beta}(i+1)\right\}dx$$
$$+2\lambda\sum_{i=0}^{\infty}\binom{2\phi-1}{i}\phi\beta\gamma(-1)^i \int_0^\infty x^{k-\beta-1}\, exp\left\{-\frac{\alpha}{x}(i+1)-\gamma\left(\frac{1}{x}\right)^{\beta}(i+1)\right\}dx,$$

Hence, we obtain the final result

$$\acute{\mu}_k = (1-\lambda)\sum_{i,j=0}^{\infty}\binom{\phi-1}{i}\frac{\gamma^j\phi(-1)^{i+j}(i+1)^j}{j!}\xi(\alpha,\beta,\gamma,i,j,k)$$
$$+2\lambda\sum_{i,j=0}^{\infty}\binom{2\phi-1}{i}\frac{\gamma^j\phi(-1)^{i+j}(i+1)^j}{j!}\xi(\alpha,\beta,\gamma,i,j,k). \qquad (13)$$

Theorem 2: If X has the $TNGIW(x;\alpha,\beta,\gamma,\phi,\lambda)$ distribution with $|\lambda|\leq 1$, then the moment generating function of X say $M_x(t)$ is given as follows

$$M_x(t)=(1-\lambda)\sum_{m=0}^{\infty}\sum_{n,\ell=0}^{\infty}\binom{\phi-1}{n}\frac{\gamma^{\ell}\phi(-1)^{n+\ell}(n+1)^{\ell}\{\alpha(n+1)t\}^{m}}{\ell!\,m!}\varrho(n,\ell,m)$$
$$+2\lambda\sum_{m=0}^{\infty}\sum_{n,\ell=0}^{\infty}\binom{2\phi-1}{n}\frac{\gamma^{\ell}\phi(-1)^{n+\ell}(n+1)^{\ell}\{\alpha(n+1)t\}^{m}}{\ell!\,m!}\varrho(n,\ell,m),$$

where

$$\varrho(n,\ell,m)=\frac{\alpha\Gamma(\beta\ell-m+1)}{\left(\alpha(n+1)\right)^{\beta\ell+1}}+\frac{\beta\gamma\Gamma(\beta(\ell+1)-m)}{\left(\alpha(n+1)\right)^{\beta(\ell+1)}}.$$

Proof: By definition the moment generating function of X can be obtained from (5) as

$$M_x(t)=$$
$$\int_0^{\infty}e^{tx}\,\phi\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}\mathcal{U}(x)dx,$$

By using equation (6) the above integral reduces to

$$M_x(t)=(1-\lambda)\int_0^{\infty}\left(\frac{1}{x}\right)^{2}\phi\alpha\,exp\left\{tx-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}dx$$
$$+(1-\lambda)\int_0^{\infty}\left(\frac{1}{x}\right)^{\beta+1}\phi\beta\gamma\,exp\left\{tx-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1\right.$$
$$\left.-\,exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1}dx$$
$$+2\lambda\int_0^{\infty}\left(\frac{1}{x}\right)^{2}\phi\alpha\,exp\left\{tx-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{2\phi-1}dx$$
$$+2\lambda\int_0^{\infty}\left(\frac{1}{x}\right)^{\beta+1}\phi\beta\gamma exp\left\{tx-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{2\phi-1}dx,$$

Finally we obtain the moment generating function of the TNGIW distribution as

$$M_x(t)=(1-\lambda)\sum_{m=0}^{\infty}\sum_{n,\ell=0}^{\infty}\binom{\phi-1}{n}\frac{\gamma^{\ell}\phi(-1)^{n+\ell}(n+1)^{\ell}\{\alpha(n+1)t\}^{m}}{\ell!\,m!}\varrho(n,\ell,m)$$
$$+2\lambda\sum_{m=0}^{\infty}\sum_{n,\ell=0}^{\infty}\binom{2\phi-1}{n}\frac{\gamma^{\ell}\phi(-1)^{n+\ell}(n+1)^{\ell}\{\alpha(n+1)t\}^{m}}{\ell!\,m!}\varrho(n,\ell,m).\ (14)$$

4. Maximum Likelihood Estimation

Consider the random samples $x_1, x_2, \dots, x_n$ consisting of n observations from the TNGIW$(x;\alpha,\beta,\gamma,\phi,\lambda)$ distribution. The log-likelihood function $\mathcal{L} = \text{lnL}$ of the density (5) for the parameter vector $\Theta = (\alpha,\beta,\gamma,\phi,\lambda)$ is given by

$$\mathcal{L} = nln\phi + \sum_{i=1}^{n} ln\left\{\alpha + \beta\gamma\left(\frac{1}{x_i}\right)^{\beta-1}\right\} - \sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right) + (\phi-1)\sum_{i=1}^{n} ln\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]$$
$$+\sum_{i=1}^{n} ln\left(\frac{1}{x_i}\right)^2 - \gamma\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta} + \sum_{i=1}^{n} ln\left\{1 - \lambda + 2\lambda\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}. \quad (15)$$

The components of score vector can be obtained by differentiating (15) with respect to α,β,γ,ϕ and λ then equating it to zero, we obtain the estimating equations are

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \sum_{i=1}^{n}\left\{\alpha + \beta\gamma\left(\frac{1}{x_i}\right)^{\beta-1}\right\}^{-1} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right) + (\phi-1)\sum_{i=1}^{n}\frac{\frac{1}{x_i} exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]}$$
$$+\sum_{i=1}^{n}\frac{2\lambda\frac{\phi}{x_i}\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi-1} exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left\{1 - \lambda + 2\lambda\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \sum_{i=1}^{n}\frac{\gamma\left(\frac{1}{x_i}\right)^{\beta-1}\left[\beta ln\left(\frac{1}{x_i}\right) + 1\right]}{\left\{\alpha + \beta\gamma\left(\frac{1}{x_i}\right)^{\beta-1}\right\}} + (\phi-1)\sum_{i=1}^{n}\frac{\gamma\left(\frac{1}{x_i}\right)^{\beta} ln\left(\frac{1}{x_i}\right) exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]}$$
$$-\gamma\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta} ln\left(\frac{1}{x_i}\right) + \sum_{i=1}^{n}\frac{2\lambda\gamma\frac{\phi}{x_i{}^{\beta}}\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi-1} exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left[ln\left(\frac{1}{x_i}\right)\right]^{-1}\left\{1 - \lambda + 2\lambda\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\gamma} = \sum_{i=0}^{n}\frac{\beta\left(\frac{1}{x_i}\right)^{\beta-1}}{\left\{\alpha + \beta\gamma\left(\frac{1}{x_i}\right)^{\beta-1}\right\}} - \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^{\beta} + (\phi-1)\sum_{i=1}^{n}\frac{\left(\frac{1}{x_i}\right)^{\beta} exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]}$$
$$+\sum_{i=1}^{n}\frac{2\lambda\frac{\phi}{x_i{}^{\beta}}\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi-1} exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}}{\left\{1 - \lambda + 2\lambda\left[1 - exp\left\{-\frac{\alpha}{x_i} - \gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\phi}=\frac{n}{\phi}+\sum_{i=1}^{n}ln\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]$$

$$+2\lambda\sum_{i=1}^{n}\frac{\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}ln\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]}{\left\{1-\lambda+2\lambda\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}},$$

$$\frac{\partial\mathcal{L}}{\partial\lambda}=\sum_{i=1}^{n}\frac{-1+2\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}}{\left\{1-\lambda+2\lambda\left[1-exp\left\{-\frac{\alpha}{x_i}-\gamma\left(\frac{1}{x_i}\right)^{\beta}\right\}\right]^{\phi}\right\}},$$

The log-likelihood function can be maximized by using the BFGS method in R or SAS languages. These nonlinear system of equations cannot be solved analytically and statistical software can be used to solve them numerically such as R-Package (Adequacy Model), SAS (PROC NLMIXED) by using iterative methods such as Limited-Memory quasi-Newton algorithm for Bound-constrained optimization (L-BFGS-B), these solutions will yield the ML estimators $\hat{\alpha},\hat{\beta},\hat{\gamma},\hat{\phi}$ and $\hat{\lambda}$. For the five parameters TNGIW distribution pdf all the second order derivatives exist. Thus we have the inverse dispersion matrix as

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\gamma}\\ \hat{\phi}\\ \hat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \phi\\ \lambda\end{pmatrix},\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} & \hat{V}_{15}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} & \hat{V}_{25}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} & \hat{V}_{35}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} & \hat{V}_{45}\\ \hat{V}_{51} & \hat{V}_{52} & \hat{V}_{53} & \hat{V}_{54} & \hat{V}_{55}\end{pmatrix}\right],$$

$$\mathrm{V}^{-1}=-\mathrm{E}\begin{bmatrix}\frac{\partial^2\mathcal{L}}{\partial\alpha^2} & \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\beta} & \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\gamma} & \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\beta} & \frac{\partial^2\mathcal{L}}{\partial\beta^2} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\gamma} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\gamma} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\gamma} & \frac{\partial^2\mathcal{L}}{\partial\gamma^2} & \frac{\partial^2\mathcal{L}}{\partial\gamma\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\gamma\,\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\gamma\,\partial\phi} & \frac{\partial^2\mathcal{L}}{\partial\phi^2} & \frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\lambda}\\ \frac{\partial^2\mathcal{L}}{\partial\alpha\,\partial\lambda} & \frac{\partial^2\mathcal{L}}{\partial\beta\,\partial\lambda} & \frac{\partial^2\mathcal{L}}{\partial\gamma\,\partial\lambda} & \frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\lambda} & \frac{\partial^2\mathcal{L}}{\partial\lambda^2}\end{bmatrix} \tag{16}$$

Equation (16) is the variance covariance matrix of the TNGIW$(x;\alpha,\beta,\gamma,\phi,\lambda)$ distribution. The asymptotic multivariate normal $N_5(0,V(\Theta)^{-1})$ distribution can be used to construct the approximate confidence intervals and confidence region of individual parameters for the transmuted new generalized inverse Weibull distribution. By using the observed information matrix an approximately $100(1-\xi)\%$ confidence intervals for α,β,γ,ϕ and λ can be determined as

$$\hat{\alpha}\pm Z_{\frac{\xi}{2}}\sqrt{\hat{V}_{11}},\qquad \hat{\beta}\pm Z_{\frac{\xi}{2}}\sqrt{\hat{V}_{22}},\qquad \hat{\gamma}\pm Z_{\frac{\xi}{2}}\sqrt{\hat{V}_{33}},\qquad \hat{\phi}\pm Z_{\frac{\xi}{2}}\sqrt{\hat{V}_{44}},\qquad \hat{\lambda}\pm Z_{\frac{\xi}{2}}\sqrt{\hat{V}_{55}}$$

where $Z_{\frac{\xi}{2}}$ is the upper ξth percentile of the standard normal distribution.

5. Entropy and Mean Deviation

The entropy is the measure of variation or the uncertainty of a random variable X for the probability density function from the lifetime distribution. The Rényi entropy for the random variable X with $f(x)$ is defined as

$$I_R(\rho)=\frac{1}{1-\rho}log\{\int f(x)^{\rho}\,dx\}, \tag{17}$$

where $\rho>0$ and $\rho\neq 1$. The integral in $I_R(\rho)$ of the TNGIW distribution can be defined as

$$\int_0^{\infty} f(x)^{\rho}dx=\int_0^{\infty}\phi^{\rho}\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)^{\rho}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\rho(\phi-1)}$$
$$\times\left(\frac{1}{x}\right)^{2\rho}exp\left\{-\frac{\alpha\rho}{x}-\gamma\rho\left(\frac{1}{x}\right)^{\beta}\right\}\left\{1-\lambda+2\lambda\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}^{\rho}dx,$$

By using the Binomial expansion, the above integral can be written as

$$=\sum_{g,\hbar=0}^{\infty}\mathcal{J}_{\phi,\rho,\lambda,g,\hbar}\int_0^{\infty}\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)^{\rho}\left(\frac{1}{x}\right)^{2\rho}exp\left\{-\frac{\alpha}{x}(\rho+\hbar)-\gamma\left(\frac{1}{x}\right)^{\beta}(\rho+\hbar)\right\}dx, \tag{18}$$

where

$$\mathcal{J}_{\phi,\rho,\lambda,g,\hbar}=\phi^{\rho}\binom{\rho}{g}\binom{\rho(\phi-1)+\phi g}{\hbar}\left(\frac{2\lambda}{1-\lambda}\right)^{g}(-1)^{\hbar}(1-\lambda)^{\rho}.$$

The above integral reduces to

$$=\sum_{g,\hbar,j=0}^{\infty}\mathcal{J}_{\phi,\rho,\lambda,g,\hbar}\binom{\rho}{j}\left(\frac{\beta\gamma}{\alpha}\right)^{j}\alpha^{\rho}\int_0^{\infty}\left(\frac{1}{x}\right)^{j(\beta-1)+2\rho}exp\left\{-\frac{\alpha}{x}(\rho+\hbar)-\gamma\left(\frac{1}{x}\right)^{\beta}(\rho+\hbar)\right\}dx, \tag{19}$$

Finally, we obtain the Rényi entropy as

$$I_R(\rho)=\frac{\rho}{1-\rho}log(\alpha)+\frac{\rho}{1-\rho}log(\phi)+\frac{\rho}{1-\rho}log(1-\lambda)+\frac{1}{1-\rho}log$$
$$\left\{\sum_{g,\hbar=0}^{\infty}\sum_{j,\mathcal{K}=0}^{\infty}\binom{\rho}{g}\binom{\rho(\phi-1)+\phi g}{\hbar}\binom{\rho}{j}\left(\frac{2\lambda}{1-\lambda}\right)^{g}\frac{(-1)^{\hbar+\mathcal{K}}}{\mathcal{K}!}\mathcal{T}(\alpha,\beta,\gamma,\rho,\hbar,j,\mathcal{K})\right\}, \tag{20}$$

where

$$\mathcal{T}(\alpha,\beta,\gamma,\rho,\hbar,j,\mathcal{K})=\frac{\gamma^{\mathcal{K}}(\rho+\hbar)^{\mathcal{K}}}{[\alpha(\rho+\hbar)]^{j(\beta-1)+\beta\mathcal{K}+2\rho-1}}\left(\frac{\beta\gamma}{\alpha}\right)^{j}\Gamma(j(\beta-1)+\beta\mathcal{K}+2\rho-1).$$

The extent of dissemination in a population is measured by the totality of deviations from the mean and the median. If X has the TNGIW$(x;\alpha,\beta,\gamma,\phi,\lambda)$, then we can derive the mean deviation about mean and about the median M can be obtain from the following equations

$$\delta_1=2\{\mu\,F(\mu)-\psi(\mu)\}\quad\text{and}\quad\delta_2=\mu-2\psi(\text{M}). \tag{21}$$

The mean is obtained from (13) with $k = 1$ and the median M is the solution of the non-linear equation is obtained from (12), where $\psi(q)$ can be attained from (5)

$$\psi(q) = (1-\lambda)\sum_{m,n=0}^{\infty}\binom{\phi-1}{m}\frac{\gamma^n\phi(-1)^{m+n}(m+1)^n}{n!\,[\alpha(m+1)]^{\beta n}}\mathcal{H}_{m,n}$$

$$+2\lambda\sum_{m,n=0}^{\infty}\binom{2\phi-1}{m}\frac{\gamma^n\phi(-1)^{m+n}(m+1)^n}{n![\alpha(m+1)]^{\beta n+\beta-1}}\,\mathcal{H}_{m,n}, \qquad (22)$$

where

$$\mathcal{H}_{m,n} = \alpha\acute{\gamma}\left\{\beta n, \frac{\alpha}{q}(m+1)\right\} + (\beta\gamma)\acute{\gamma}\left\{\beta n+\beta-1, \frac{\alpha}{q}(m+1)\right\}.$$

Hence, the measure in (21) can be obtained from (22). The quantity $\psi(q)$ can also be used to determine the Bonferroni and the Lorenz curves which have applications in econometrics and finance. They are given by

$$B(P) = \frac{\psi(q)}{P\mu} \qquad \text{and} \qquad L(P) = \frac{\psi(q)}{\mu},$$

Where $q = Q(P)$ is calculated from (12) for a given probability P.

6. Order Statistics

Let $x_1, x_2, \ldots, x_n$ are independently identically distributed ordered random variables from the TNGIW distribution then the pdf of rth order statistic $x_{(r)}$ is given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\sum_{\mathcal{P}=0}^{n-r}\binom{n-r}{\mathcal{P}}(-1)^{\mathcal{P}}\big(F(x)\big)^{r+\mathcal{P}-1}f(x) \qquad (23)$$

where B(.,.) is the beta function, by substituting (5) and (7) into (23) we obtain

$$f_{r:n}(x) = n\binom{n-1}{r-1}\sum_{\mathcal{P}=0}^{n-r}\sum_{\mathcal{Q}=0}^{\infty}\binom{n-r}{\mathcal{P}}\binom{r+\mathcal{P}-1}{\mathcal{Q}}(-1)^{\mathcal{P}}\left[1 - exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi(\mathcal{Q}+1)-1}$$

$$\times\,\phi\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)\left\{1+\lambda\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi}\right\}^{r+\mathcal{P}-1}\exp\left\{-\frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta}\right\}\mathcal{U}(x)$$

$$f_{r:n}(x) = n\binom{n-1}{r-1}\sum_{\mathcal{P}=0}^{n-r}\sum_{\mathcal{Q},\mathcal{S}=0}^{\infty}\psi_{\mathcal{P},\mathcal{Q},\mathcal{S},\lambda}\;\mathcal{T}(\mathcal{Q},\mathcal{S}), \qquad (24)$$

where

$$\psi_{\mathcal{P},\mathcal{Q},\mathcal{S},\lambda} = \binom{n-r}{\mathcal{P}}\binom{r+\mathcal{P}-1}{\mathcal{Q}}\binom{r+\mathcal{P}-1}{\mathcal{S}}(-1)^{\mathcal{P}+\mathcal{S}}\lambda^{\mathcal{S}}\phi,$$

and

$$\mathcal{T}(\mathcal{Q},\mathcal{S})=\left(\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right)\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-\exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi(\mathcal{Q}+\mathcal{S}+1)-1}\mathcal{U}(\mathrm{x}).$$

Using (24), the *S*th moment of *r*th order statistic of $x_{(r)}$is given by

$$\mu_s^{(r:n)}=n\binom{n-1}{r-1}\sum_{\mathcal{P}=0}^{n-r}\sum_{\mathcal{Q},\mathcal{S}=0}^{\infty}\psi_{\mathcal{P},\mathcal{Q},\mathcal{S},\lambda}\ \left\{(1-\lambda)\mathcal{E}_{\mathcal{Q},\mathcal{S},\mathcal{K},\mathcal{j},1}\mathrm{V}_{\mathcal{K},\mathcal{j}}+2\lambda\mathcal{E}_{\mathcal{Q},\mathcal{S},\mathcal{K},\mathcal{j},2}\mathrm{V}_{\mathcal{K},\mathcal{j}}\right\},\quad(25)$$

where

$$\mathcal{E}_{\mathcal{Q},\mathcal{S},\mathcal{K},\mathcal{j},g}=\sum_{\mathcal{K},\mathcal{j}=0}^{\infty}\binom{\phi(\mathcal{Q}+\mathcal{S}+g)-1}{\mathcal{K}}\frac{(-1)^{\mathcal{K}+\mathcal{j}}[\gamma(\mathcal{K}+1)]^{\mathcal{j}}}{\mathcal{j}!},g=1,2$$

$$\mathrm{V}_{\mathcal{K},\mathcal{j}}=\frac{\alpha\Gamma(\beta\mathcal{j}-\mathcal{K}+1)}{\{\alpha(\mathcal{K}+1)\}^{(\beta\mathcal{j}-\mathcal{K}+1)}}+\frac{\beta\gamma\Gamma(\beta(\mathcal{j}+1)-\mathcal{K})}{\{\alpha(\mathcal{K}+1)\}^{(\beta(\mathcal{j}+1)-\mathcal{K})}}$$

7. Applications

In this section, we illustrate the usefulness of the TNGIW distribution to two real data sets.

7.1. *Application 1: Ball bearings data*

The first subsection provides the data analysis in order to assess the goodness-of-fit of the proposed model with failure times. We consider the ball bearings data for the number of revolution before failure, each of 23 ball bearings in the life tests are as follows
17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

The data set is reported by Lawless (1982). Transmuted new generalized inverse Weibull (TNGIW), new generalized inverse Weibull (NGIW), Kumaraswamy modified inverse Weibull (KMIW), Exponentiated Kumaraswamy inverse Weibull (EKIW), Kumaraswamy inverse weibull (KIW) and modified inverse Weibull (MIW) distributions are fitted to the ball bearings data.

(1) New generalized inverse Weibull (NGIW) distribution with the pdf

$$f(x)=\phi\left\{\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right\}exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\left[1-exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}\right]^{\phi-1},x>0,$$

where $\alpha,\gamma>0$ are the scale parameters and $\beta,\phi>0$ is the shape parameter of the NGIW distribution. (Khan and King, 2016)

(2) Kumaraswamy Modified inverse Weibull (KMIW) distribution with the pdf

$$f(x)=\phi\lambda\left\{\frac{\alpha}{x^2}+\frac{\beta\gamma}{x^{\beta+1}}\right\}exp\left\{-\lambda\left(\frac{\alpha}{x}+\frac{\gamma}{x^{\beta}}\right)\right\}\left[1-exp\left\{-\lambda\left(\frac{\alpha}{x}+\frac{\gamma}{x^{\beta}}\right)\right\}\right]^{\phi-1},x>0,$$

where $\alpha,\gamma,\lambda>0$ are the scale parameters and $\beta,\phi>0$ is the shape parameter of the KMIW distribution. (Aryal and Elbatal, 2015)

(3) Exponentiated Kumaraswamy inverse Weibull (EKIW) distribution with the pdf

$$f(x) = \beta\lambda\gamma\phi\alpha^{\beta}x^{-(\beta+1)}\,exp\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)\left\{1-exp\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)\right\}^{\gamma-1},$$
$$\left\{1-\left(1-exp\left(-\lambda\left(\frac{\alpha}{x}\right)^{\beta}\right)\right)^{\gamma}\right\}^{\phi}, \qquad x>0,$$

where $\beta, \gamma, \phi > 0$ are the shape parameters and $\alpha, \lambda > 0$ is the scale parameter of the EKIW distribution. (Rodrigues. et al. 2016)

(4) Kumaraswamy inverse Weibull (KIW) distribution with the pdf

$$f(x) = \alpha\beta\gamma\phi\left(\frac{1}{x}\right)^{\phi+1}exp\left(-\alpha\gamma\left(\frac{1}{x}\right)^{\phi}\right)\left\{1-exp\left(-\alpha\gamma\left(\frac{1}{x}\right)^{\phi}\right)\right\}^{\beta-1}, \qquad x>0,$$

where $\beta, \phi > 0$ are the shape parameters and $\alpha, \gamma > 0$ is the scale parameter of the KIW distribution. (Shahbaz, et al. 2012)

(5) Modified inverse Weibull (MIW) distribution with the pdf

$$f(x) = \left\{\alpha+\beta\gamma\left(\frac{1}{x}\right)^{\beta-1}\right\}\left(\frac{1}{x}\right)^{2}exp\left\{-\frac{\alpha}{x}-\gamma\left(\frac{1}{x}\right)^{\beta}\right\}, \qquad x>0,$$

where $\alpha, \theta > 0$ are the scale parameters and $\eta > 0$ is the shape parameter of the MIW distribution. (Khan and King, 2012)

Table 2: MLEs of the Parameters for ball bearings data

Model	Parameter Estimates				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$
TNGIW	38.9435 (116.66)	0.3037 (0.4461)	19.2623 (31.457)	269.05 (1502.4)	0.0830 (1.5102)
KMIW	0.0033 (59.1497)	0.4652 (0.8550)	12.2704 (10.9417)	81.0349 (518.868)	2.7148 (15.7467)
EKIW	15.19578 (199.86)	0.6798 (4.0056)	38.9055 (391.486)	0.5590 (5.6098)	12.8083 (95.0540)
NGIW	28.7722 (106.97)	0.3125 (0.4431)	20.7116 (27.708)	294.19 (1633.9)	-
KIW	4.9550 (33.011)	81.1513 (363.29)	6.7215 (44.781)	0.4650 (0.4365)	-
MIW	0.0011 (42.172)	1.8341 (0.3534)	1240.1 (1249.9)	-	-

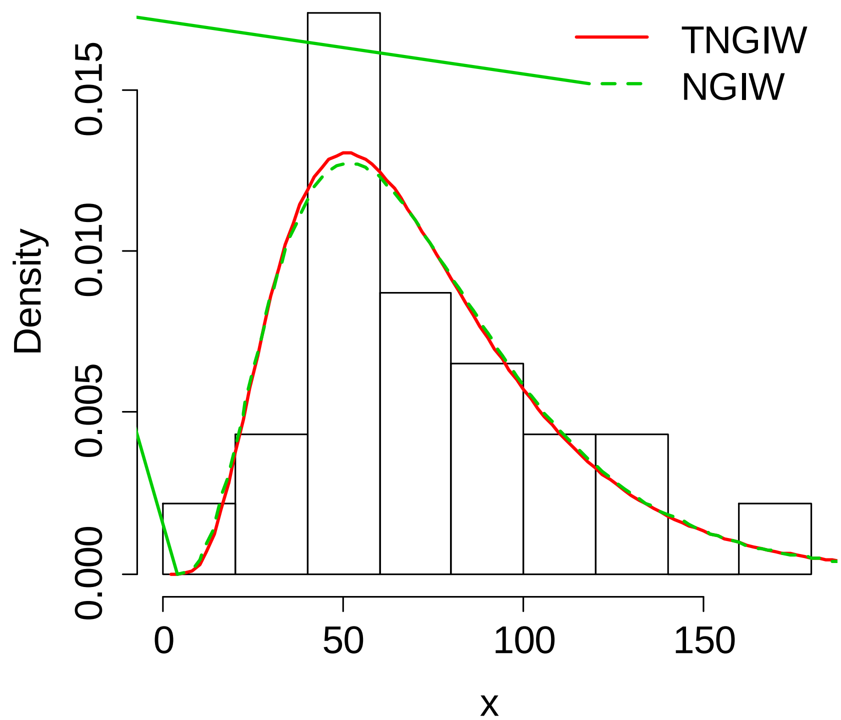

Figure 3: Fitted Models for failure of ball bearings data

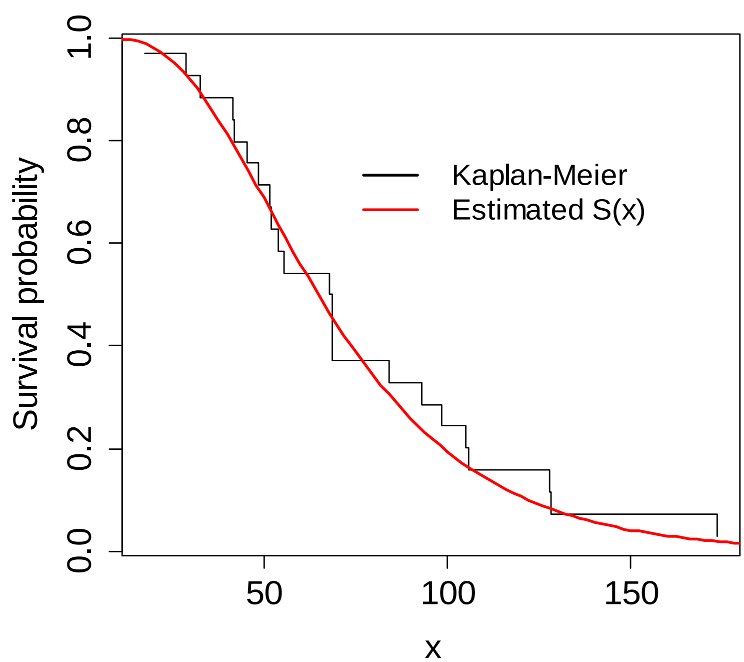

Figure 4: Estimated Survival function for the TNGIWD for ball bearings data

Table 2 listed the MLEs of the unknown parameter(s) and the corresponding standard errors for the model parameters. In order to evaluate the performance of the TNGIW distribution and can be consider as a superior lifetime model, we shall compare the goodness of fit with five other lifetime distributions recently proposed in the literature. The visualization of the estimated densities with histogram displayed in Figure 3 indicate that the transmuted new generalized inverse Weibull distribution has the better estimates comparing with other five distributions. Hence the data points from the TNGIW distribution has better relationship and can be consider as the virtuous model for life time data. For the goodness of fit statistics, we use the Kolmogorov-Smirnov (K-S) test to see which model provides the better estimates and results are displayed in Table 2. In order to assess if the model is appropriate, Figure 4 plots the empirical and estimated survival functions of the TNGIW distribution.

Table 3: Cramér-von Mises, Anderson-Darling goodness of-fit statistics and K-S Test

Distribution	$\mathcal{W}$	$\mathcal{A}$	K-S Test
TNGIW	0.0318	0.1903	0.1061
KMIW	0.0338	0.1954	0.1115
EKIW	0.0346	0.2007	0.1149
NGIW	0.0335	0.1953	0.1105
KIW	0.0337	0.1955	0.1114
MIW	0.0752	0.5552	0.1328

To further verify which distribution provides the better estimates for ball bearings data, we apply the Cramér-von Mises and Anderson-Darling goodness of-fit statistics and results are displayed in Table 3. The smaller values of these statistics indicate the better fit. We detect from Tables 2 and 3 that the TNGIW distribution has the lowest values for the Kolmogorov-Smirnov (K-S) test, Cramér-von Mises and Anderson-Darling goodness of-fit statistics among the all fitted distributions recently proposed in the literature. Therefore the TNGIW distribution can be consider as a good model for the failure times of ball bearings data. Figures 4 displays the estimated Survival function of the TNGIW distribution with better relationship for the ball bearings data.

7.2. *Application 2: Fatigue life of aluminium data*

The second data set is prearranged by Birnbaum and Saunders (1969) on the fatigue life of 6061-T6 aluminium coupons cut parallel with the course of rolling and oscillated at 18 cycles per second. The data set comprises of 101 observations with maximum stress per cycle 31,000 psi. The data are

70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109,112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130,130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142,142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163,163, 164, 166, 166, 168, 170, 174, 201, 212.

Table 4: MLEs of the Parameters for fatigue life of aluminium data and AIC

Model	Parameter Estimates				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\phi}$	$\hat{\lambda}$
TNGIW	765.69	1.4084	927.14	313.48	0.8173
	(933.96)	(1.1478)	(1753.23)	(389.19)	(0.1910)
KMIW	55.5137	143.61	1.4551	66.9352	5.5750
	(199.86)	(54.005)	(0.7554)	(173.34)	(95.054)
EKIW	33.6575	1.0858	1.6386	121.7002	21.0374
	(65.1637)	(0.9365)	(1.6291)	(328.69)	(0.8763)
NGIW	686.03	1.3005	649.13	362.14	-
	(693.55)	(0.6479)	(1844.21)	(361.51)	
KIW	71.1614	64.5064	90.4453	1.4842	-
	(306.61)	(98.419)	(389.55)	(0.5042)	
MIW	0.0001	3.0354	2.24E+6	-	-
	(79.754)	(0.0557)	(1350.4)		

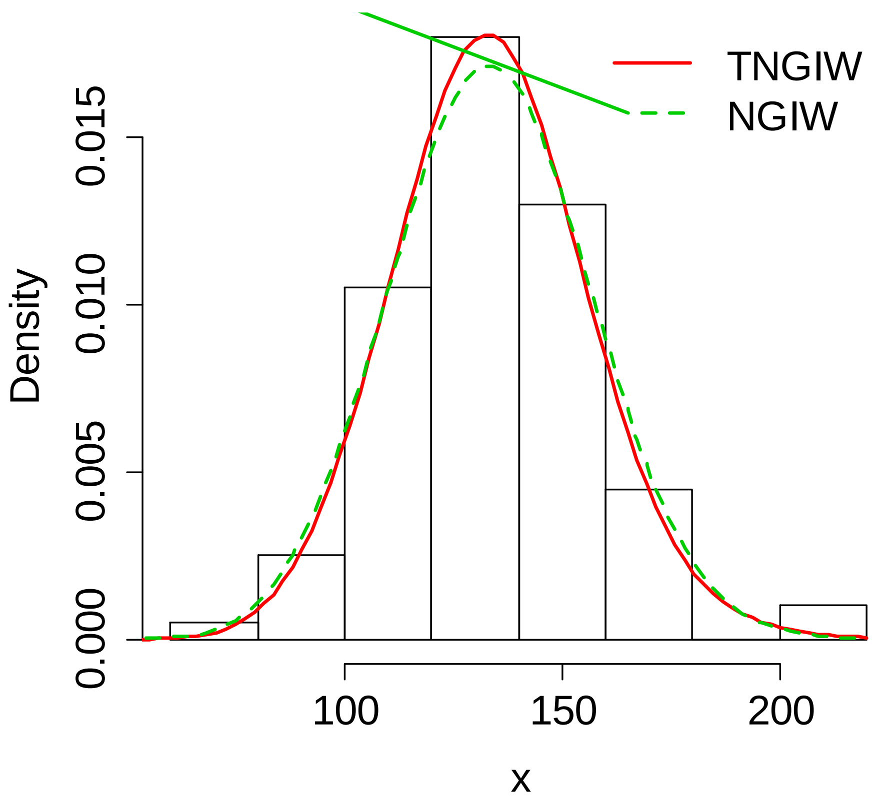

Figure 5: Fitted Models for failure of fatigue life aluminium data

Table 5: Cramér-von Mises, Anderson-Darling goodness of-fit statistics and K-S Test

Distribution	$\mathcal{W}$	$\mathcal{A}$	K-S Test
TNGIW	0.0416	0.2893	0.0574
KMIW	0.0495	0.3507	0.0604
EKIW	0.0506	0.3376	0.0658
NGIW	0.0531	0.3830	0.0662
KIW	0.0496	0.3424	0.0639
MIW	0.2274	1.3069	0.2863

We examine the use of the use of the Transmuted new generalized inverse Weibull (TNGIW) distribution for modelling the fatigue fracture life of aluminium data. We fitted the TNGIW, NGIW, KMIW, EKIW, KIW and MIW densities are displayed in Table 4 and goodness of fit measures are listed in Table 5. The histogram of the fatigue fracture life of aluminium data is shown in Figure 5 along with the estimated densities of the TNGIW, NGIW models. The fitted model suggest that the TNGIW distribution is reasonable. These goodness of fit results indicate that the TNGIW distribution has the lowest values of the Kolmogorov-Smirnov (K-S) test, Cramér-von Mises and Anderson-Darling goodness of-fit statistics among the all fitted distributions. We conclude that the TNGIW distribution provides a good fit to these data sets.

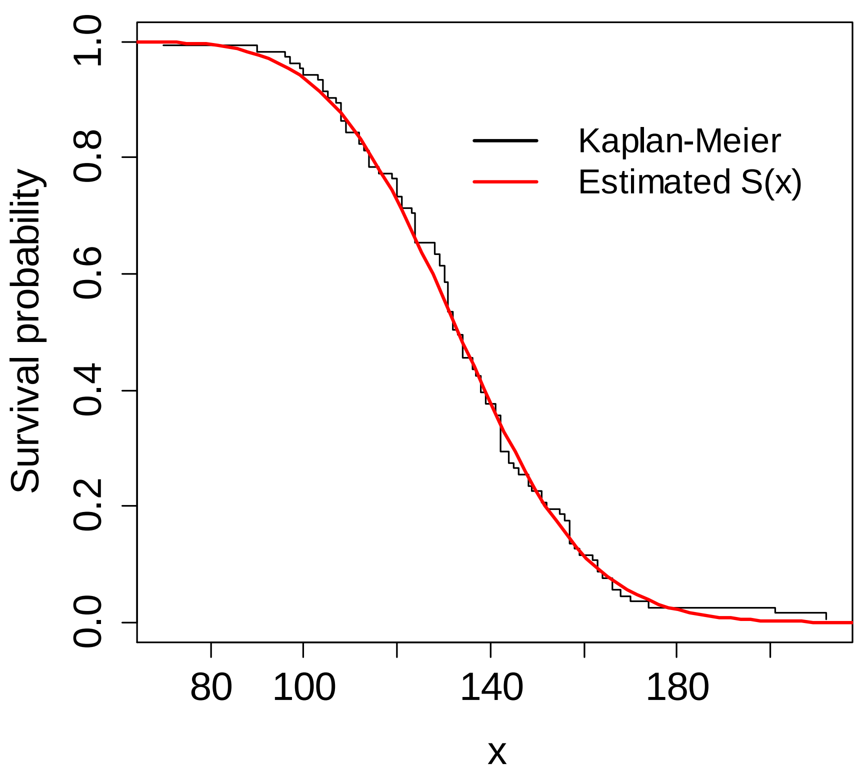

Figure 6: Estimated Survival function of TNGIWD for fatigue life aluminium data

8. Concluding Remarks

We studied and formulated some theoretical properties of the new distribution called the transmuted new generalized inverse Weibull distribution. The new distribution presents a generalization of several models previously considered in the literature such as transmuted modified inverse Weibull distribution, transmuted generalized inverse Weibull distribution, transmuted inverse Weibull distribution, modified inverse Weibull distribution. The proposed distribution has twenty three lifetime distributions as special cases. This proposed model has the upside down bathtub shape failure rate patterns. The method of maximum likelihood is employed for estimating the model parameters. The usefulness of the new model is illustrated in two applications. We have anticipation that the proposed model may attract wider applications in the analysis of lifetime data.

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